

Entanglement Entropy of Coupled Conformal Field Theories and Fermi Liquids

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In this paper we calculate the entanglement Rényi entropy of two coupled gapless systems in general spatial dimension d . The gapless systems can be either conformal field theories (CFT), or Fermi liquids. We assume the two systems are coupled uniformly in a h -dimensional submanifold of the space, with $0 \leq h \leq d$. We will focus on the scaling of the Rényi entropy with the size of the system, and its scaling with the inter-system coupling constant g . Three approaches will be used for our calculation: (1) exact calculation with ground state wave-functional, (2) perturbative calculation with functional path integral, (3) scaling argument.

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I. DEFINE THE PROBLEM

The entanglement entropy of a system \mathcal{H} , refers to the entropy of the reduced density matrix of subsystem \mathcal{A} , after tracing out the rest of the system \mathcal{H}/\mathcal{A} . Usually \mathcal{A} and \mathcal{H}/\mathcal{A} are simply separated spatially. If \mathcal{H} is a conformal field theory (CFT), it is well-understood that for dimension $d > 1$, the leading contribution to the entanglement entropy $S_{\mathcal{A}}$ is proportional to the size of the boundary of subsystem \mathcal{A}^1 ; while at $d = 1$, $S_{\mathcal{A}}$ is proportional to $\ln(L_{\mathcal{A}})$, $L_{\mathcal{A}}$ is the size of \mathcal{A}^{2-4} . If \mathcal{H} is a Fermi liquid, then the leading order entanglement entropy is $S_{\mathcal{A}} \sim L_{\mathcal{A}}^{d-1} \log(L_{\mathcal{A}})$ with arbitrary dimension d^{5-7} .

In this work, we study the entanglement entropy between two coupled systems \mathcal{A} and \mathcal{B} in general dimension d , namely the entanglement entropy of the reduced density matrix $\rho_{\mathcal{A}}$, after tracing out \mathcal{B} . The entire action of the system can be schematically written as

$$\mathcal{S} = \int d\tau d^d x \mathcal{L}_{\mathcal{A}} + \mathcal{L}_{\mathcal{B}} + g O_{\mathcal{A}}(x) O_{\mathcal{B}}(x) V(x). \quad (1)$$

$O_{\mathcal{A}}$ and $O_{\mathcal{B}}$ are normal ordered operators ($\langle O_{\mathcal{A}} \rangle = \langle O_{\mathcal{B}} \rangle = 0$) of \mathcal{A} and \mathcal{B} respectively, $V(x)$ is a function of space. The entanglement entropy is in general a complicated function of g (even for most of the exactly solvable cases), so we will always assume a weak coupling g , and focus on the scaling of the entropy with weak g . Our current work will focus on the Rényi entropy, which is defined as

$$S_{\mathcal{A},n} = \frac{1}{1-n} \log(\text{tr} \rho_{\mathcal{A}}^n). \quad (2)$$

We take $n = 2$ as an example hereafter, but the results of our paper are insensitive to n .

Our system is defined on a d -dimensional space with coordinates (x_1, \dots, x_d) , and $-L < x_i < L$. We assume \mathcal{A} and \mathcal{B} are coupled uniformly in a h -dimensional submanifold \mathcal{M} of the d dimensional space, *i.e.*

$$V(x_1, \dots, x_h, 0, \dots, 0) = 1, \quad V(x) = 0 \text{ otherwise} \quad (3)$$

When $h = d$, the coupling is uniform in the entire space, while when $h = 0$ the coupling is only through a point contact.

This paper is outlined as follows: In section II we will introduce a general functional path integral and perturbation theory for the entanglement entropy, and it will be applied to the coupled Fermi liquids in section III. In section IV, we will study the entanglement entropy of coupled CFTs. Besides the perturbation theory, we will also introduce a scaling argument to understand the qualitative behavior of the entanglement entropy. The results with $h = d$ will also be checked with exact calculations based on the ground state wave functionals of the coupled CFTs.

II. FUNCTIONAL PATH INTEGRAL AND PERTURBATION THEORY

In most cases, the entanglement entropy between coupled systems cannot be calculated exactly. In this section we will introduce a general perturbation theory based on functional path integral formalism for the entanglement entropy.

First of all, the reduced density matrix $\rho_{\mathcal{A}}$ of subsystem \mathcal{A} at zero temperature is

$$\begin{aligned} \langle \varphi_{\mathcal{A}} | \rho_{\mathcal{A}} | \varphi'_{\mathcal{A}} \rangle &= \text{tr}_{\mathcal{B}} [\rho_{\mathcal{AB}}] \\ &= \lim_{\beta \rightarrow \infty} \frac{1}{Z_{\mathcal{AB}}^{(\beta)}} \int D\varphi_{\mathcal{A}} D\varphi_{\mathcal{B}} |_{\varphi_{\mathcal{A}}(0)=\varphi_{\mathcal{A}}, \varphi_{\mathcal{A}}(\beta)=\varphi'_{\mathcal{A}}} \\ &\quad \times \exp \left(- \int_0^\beta d\tau d^d x \mathcal{L}(\varphi_{\mathcal{A}}, \varphi_{\mathcal{B}}) \right). \end{aligned} \quad (4)$$

$Z_{\mathcal{AB}}^{(\beta)}$ is the partition function of the entire system:

$$Z_{\mathcal{AB}}^{(\beta)} = \int D\varphi_{\mathcal{A}} D\varphi_{\mathcal{B}} \exp \left(- \int_0^\beta d\tau d^d x \mathcal{L}(\varphi_{\mathcal{A}}, \varphi_{\mathcal{B}}) \right). \quad (5)$$

In the Lagrangian, if $\varphi_{\mathcal{A}}$ and $\varphi_{\mathcal{B}}$ are boson (fermion) fields, they are periodic (antiperiodic) in imaginary time $\tau \in (0, \beta)$. In this section, $\varphi_{\mathcal{A}}$ and $\varphi_{\mathcal{B}}$ will be taken as boson fields for example.

In order to calculate the Rényi entropy with $n = 2$, we

need to evaluate the following quantity:

$$\begin{aligned} \text{tr}[\rho_{\mathcal{A}}^2] &= \lim_{\beta \rightarrow \infty} \frac{1}{Z_{\mathcal{AB}}^{(\beta)^2}} \int D\varphi_{\mathcal{A}} D\varphi_{\mathcal{B}} \\ &\times \exp\left[-\int d^d x \int_0^{2\beta} d\tau \mathcal{L}_{\mathcal{A}}(\varphi_{\mathcal{A}})\right. \\ &\left.- \left(\int_0^{\beta-\epsilon} + \int_{\beta+\epsilon}^{2\beta}\right) d\tau (\mathcal{L}_{\mathcal{B}}(\varphi_{\mathcal{B}}) + g O_{\mathcal{A}} O_{\mathcal{B}} V(x))\right]. \quad (6) \end{aligned}$$

In the numerator of this equation, we will keep

$$\begin{aligned} \varphi_{\mathcal{A}}(0) &= \varphi_{\mathcal{A}}(2\beta), \\ \varphi_{\mathcal{B}}(0) &= \varphi_{\mathcal{B}}(\beta - \epsilon), \quad \varphi_{\mathcal{B}}(\beta + \epsilon) = \varphi_{\mathcal{B}}(2\beta). \quad (7) \end{aligned}$$

ϵ is an infinitesimal positive number. Field $\varphi_{\mathcal{B}}$ is integrated out in imaginary time segments $\tau \in (0, \beta - \epsilon)$ and $\tau \in (\beta + \epsilon, 2\beta)$ separately.

The periodicity difference of $\varphi_{\mathcal{A}}(\tau)$ and $\varphi_{\mathcal{B}}(\tau)$ is the key of this calculation. Although we are always considering the case with zero temperature, the limit $\beta \rightarrow +\infty$ should be taken *after* all the calculations with finite β .

Now we try to calculate the entanglement entropy with perturbation of g . Since we assumed that $O_{\mathcal{A}}$ and $O_{\mathcal{B}}$ are both normal ordered, the first order perturbation of g of Eq. 6 vanishes. Expanding Eq. 6 to the second order of g , we obtain

$$\begin{aligned} \log(\text{tr}\rho_{\mathcal{A}}^2) &\sim \lim_{\beta \rightarrow +\infty} (S_1 - S_2), \\ S_1 &\sim g^2 \left(\int_0^{\beta-\epsilon} + \int_{\beta+\epsilon}^{2\beta} \right) d\tau_1 d\tau_2 d^d x_1 d^d x_2 V(x_1) V(x_2) \\ &\times G_{\mathcal{AA}}^{(2\beta)}(\tau_1, x_1, \tau_2, x_2) G_{\mathcal{BB}}^{(\beta)}(\tau_1, x_1, \tau_2, x_2); \\ S_2 &\sim 2g^2 \int_0^{\beta} d\tau_1 d\tau_2 d^d x_1 d^d x_2 V(x_1) V(x_2) \\ &\times G_{\mathcal{AA}}^{(\beta)}(\tau_1, x_1, \tau_2, x_2) G_{\mathcal{BB}}^{(\beta)}(\tau_1, x_1, \tau_2, x_2). \quad (8) \end{aligned}$$

$G_{\mathcal{AA}}$ and $G_{\mathcal{BB}}$ are correlation functions:

$$\begin{aligned} G_{\mathcal{AA}}(\tau_1, x_1, \tau_2, x_2) &= \langle O_{\mathcal{A}}(\tau_1, x_1) O_{\mathcal{A}}(\tau_2, x_2) \rangle, \\ G_{\mathcal{BB}}(\tau_1, x_1, \tau_2, x_2) &= \langle O_{\mathcal{B}}(\tau_1, x_1) O_{\mathcal{B}}(\tau_2, x_2) \rangle, \\ G^{(2\beta)}(\tau_1, x_1, \tau_2, x_2) &= G^{(2\beta)}(\tau_1 + 2\beta, x_1, \tau_2, x_2), \\ G^{(\beta)}(\tau_1, x_1, \tau_2, x_2) &= G^{(\beta)}(\tau_1 + \beta, x_1, \tau_2, x_2). \quad (9) \end{aligned}$$

Notice that there are two different periodicities in these correlation functions. If $O_{\mathcal{A}}$ and $O_{\mathcal{B}}$ are both bosonic operators, then in the frequency space, $G^{(\beta)}$ has Matsubara frequency $2\pi m/\beta$, while $G^{(2\beta)}$ has frequency $2\pi m/(2\beta)$. For example, if \mathcal{A} and \mathcal{B} are CFTs with $z = 1$, and we assume operator $O_{\mathcal{A}(\mathcal{B})}$ has scaling dimension $\Delta_{\mathcal{A}(\mathcal{B})}$, then $G^{(\beta)}$ reads

$$G_{\mathcal{AA}(\mathcal{BB})}^{(\beta)}(0, 0, \tau, x)$$

$$\sim \frac{1}{\beta L^d} \sum_{\omega, k} \left(\frac{1}{\omega^2 + k^2} \right)^{\frac{1}{2}(d+1-2\Delta_{\mathcal{A}(\mathcal{B})})} e^{i\omega\tau + i\vec{k}\cdot\vec{x}}. \quad (10)$$

Eq. 6 and Eq. 8 were formulated for Rényi entropy with $n = 2$ only, but their generalization to arbitrary n is straightforward.

III. ENTANGLEMENT ENTROPY OF COUPLED FERMILY LIQUIDS

A. Uniform tunnelling

In this section we will consider the entanglement entropy of coupled Fermi liquids. The simplest situation that we can start with, is that \mathcal{A} and \mathcal{B} are free Fermi gases with $S^z = \pm 1/2$ respectively, and they are coupled together through a uniform transverse magnetic field HS^x . Our goal is calculate the entanglement entropy of $S^z = 1/2$ fermions, after tracing out the $S^z = -1/2$ fermions. With uniform magnetic field, this system can be trivially solved, and the reduced density matrix $\rho_{\mathcal{A}}$ is a simple direct product of the density matrix at each momentum k :

$$\rho_{\mathcal{A}} = \prod_k \otimes \rho_{\mathcal{A},k}. \quad (11)$$

If both $S^x = \pm 1/2$ spin states are occupied or unoccupied, $\rho_{\mathcal{A},k}$ is a pure state density matrix. While if only one of the spin states is occupied, $\rho_{\mathcal{A},k}$ is maximally entangled:

$$\rho_{\mathcal{A},k} = \frac{1}{2} c_{k,\uparrow}^\dagger |0\rangle \langle 0| c_{k,\uparrow} + \frac{1}{2} |0\rangle \langle 0|. \quad (12)$$

Therefore only the states with energy $\varepsilon_f - H/2 < \varepsilon < \varepsilon_f + H/2$ contribute to the entanglement entropy. Hence the entanglement entropy should scale as

$$S_{\mathcal{A}} \sim \mathcal{N}(\varepsilon_f) |H| L^d. \quad (13)$$

$\mathcal{N}(\varepsilon_f)$ is the density of states at the Fermi surface.

B. Point contact tunnelling

Now suppose spin up and down fermions are coupled through a static polarized magnetic impurity at $\vec{r} = 0$: $HS^x(0)$, this impurity tunnels spin up and down fermions through the point contact at $\vec{r} = 0$. The perturbation formalism developed in the previous section is applicable here, as long as in the calculation we keep the Matsubara frequency for spin up (\mathcal{A}) and down (\mathcal{B}) fermions as

$$\omega_{\mathcal{A}} = \frac{\pi(2m+1)}{2\beta}, \quad \omega_{\mathcal{B}} = \frac{\pi(2n+1)}{\beta}. \quad (14)$$

Notice that the difference between Matsubara frequencies $\omega_{\mathcal{A}}$ and $\omega_{\mathcal{B}}$ leads to

$$\int_0^\beta d\tau \exp(i(\omega_{\mathcal{A}} - \omega_{\mathcal{B}})\tau) = \frac{i - (-1)^m}{\omega_{\mathcal{A}} - \omega_{\mathcal{B}}}, \quad (15)$$

which contrasts the delta function in the usual case.

The leading order contribution to the entanglement entropy is a straightforward application of Eq. 8, and it leads to the following results:

$$\begin{aligned} \log(\text{tr} \rho_{\mathcal{A}}^2) &= \lim_{\beta \rightarrow +\infty} (S'_1 - S'_2), \\ S'_1 &\sim H^2 \sum_{\omega_{\mathcal{A}}, \omega_{\mathcal{B}}} \sum_{k_{\mathcal{A}}, k_{\mathcal{B}}} \frac{2 L^{-2d}}{\beta^2 (\omega_{\mathcal{A}} - \omega_{\mathcal{B}})^2} \\ &\times \frac{1}{i\omega_{\mathcal{A}} - \varepsilon_{k,\mathcal{A}} + \varepsilon_f} \frac{1}{i\omega_{\mathcal{B}} - \varepsilon_{k,\mathcal{B}} + \varepsilon_f}; \\ S'_2 &\sim 2H^2 \sum_{\omega} \sum_{k_{\mathcal{A}}, k_{\mathcal{B}}} L^{-2d} \\ &\times \frac{1}{i\omega - \varepsilon_{k,\mathcal{A}} + \varepsilon_f} \frac{1}{i\omega - \varepsilon_{k,\mathcal{B}} + \varepsilon_f}. \end{aligned} \quad (16)$$

Frequency ω takes the usual values $\pi(2m+1)/\beta$.

Correct evaluation of the frequency and momentum summation in Eq. 16 leads to the following result:

$$\begin{aligned} S_{\mathcal{A}} &\sim H^2 \left(\int_{\varepsilon_f}^{+\infty} d\varepsilon_{\mathcal{A}} \int_0^{\varepsilon_f} d\varepsilon_{\mathcal{B}} + \int_0^{\varepsilon_f} d\varepsilon_{\mathcal{A}} \int_{\varepsilon_f}^{+\infty} d\varepsilon_{\mathcal{B}} \right) \\ &\times \frac{1}{(\varepsilon_{\mathcal{A}} - \varepsilon_{\mathcal{B}})^2} \mathcal{N}(\varepsilon_{\mathcal{A}}) \mathcal{N}(\varepsilon_{\mathcal{B}}). \end{aligned} \quad (17)$$

Since the density of states is a constant close to the Fermi surface, this integral is logarithmically divergent when $\varepsilon_{\mathcal{A}}$ and $\varepsilon_{\mathcal{B}}$ are close to Fermi energy ε_f . This logarithmic divergence will be cut-off by $1/L$, thus the final result of the entanglement entropy is

$$S_{\mathcal{A}} \sim H^2 (\mathcal{N}(\varepsilon_f))^2 \log(L). \quad (18)$$

In one dimension, the Fermi liquid becomes Luttinger liquid, which is a CFT. Uniform magnetic field and point contact single fermion tunnelling have scaling dimensions $\Delta = 1$ and 0 respectively on a free fermion Luttinger liquid CFT, *i.e.* the point contact single fermion tunnelling is a marginal perturbation on the free fermion Luttinger liquid. Later we will see that the results in Eq. 13 and Eq. 18 are consistent with our general results about CFT with $d = 1$.

Many aspects of the Fermi liquid theory can be viewed as infinite number of one dimensional fermions moving along the radial direction, thus it is not surprising that the entanglement entropy of Fermi liquid at higher dimension is qualitatively equivalent to one dimensional free fermions. The connection between the Fermi liquid and one dimensional CFT was also used to understand the ordinary entanglement entropy of Fermi liquid⁷.

C. h -dimensional tunnelling

Now Let us assume the $S^z = \pm 1/2$ fermions are coupled through a transverse magnetic field on a h -dimensional submanifold \mathcal{M} of the space (Eq. 3). If we take the simplest quadratic fermion dispersion, the second order perturbation in Eq. 8 gives the following result:

$$S_{\mathcal{A}} = S'_1 - S'_2,$$

$$\begin{aligned} S'_1 &\sim H^2 \sum_{\omega_{\mathcal{A}}, \omega_{\mathcal{B}}} \sum_{k_i, k_{\mathcal{A},j}, k_{\mathcal{B},j}} \frac{2 L^{2h-2d}}{\beta^2 (\omega_{\mathcal{A}} - \omega_{\mathcal{B}})^2} \\ &\times \frac{1}{i\omega_{\mathcal{A}} - \sum_{i=1}^h k_i^2 - \sum_{j=h+1}^d k_{\mathcal{A},j}^2 + \varepsilon_f} \\ &\times \frac{1}{i\omega_{\mathcal{B}} - \sum_{i=1}^h k_i^2 - \sum_{j=h+1}^d k_{\mathcal{B},j}^2 + \varepsilon_f}; \\ S'_2 &\sim H^2 \sum_{\omega} \sum_{k_i, k_{\mathcal{A},j}, k_{\mathcal{B},j}} L^{2h-2d} \\ &\times \frac{1}{i\omega - \sum_{i=1}^h k_i^2 - \sum_{j=h+1}^d k_{\mathcal{A},j}^2 + \varepsilon_f} \\ &\times \frac{1}{i\omega - \sum_{i=1}^h k_i^2 - \sum_{j=h+1}^d k_{\mathcal{B},j}^2 + \varepsilon_f}. \end{aligned} \quad (19)$$

When $h < d$, this integral is always logarithmically divergent, thus the logarithmic contribution persists (at least to the second order perturbation) as long as $h < d$:

$$S_{\mathcal{A}} \sim H^2 L^h \log(L). \quad (20)$$

IV. ENTANGLEMENT ENTROPY OF COUPLED CONFORMAL FIELD THEORIES

If the two coupled systems \mathcal{A} and \mathcal{B} are both CFTs, the entanglement entropy due to coupling g will obviously depend on the scaling dimension Δ of the coupling constant g . If the scaling dimensions of $O_{\mathcal{A}}$ and $O_{\mathcal{B}}$ are $\Delta_{\mathcal{A}}$ and $\Delta_{\mathcal{B}}$ respectively, then the dimension of g is $\Delta = h + z - \Delta_{\mathcal{A}} - \Delta_{\mathcal{B}}$, z is the dynamical exponent.

A. Exact calculation with ground state wave-functionals

We will first consider the following theory

$$\begin{aligned} \mathcal{L} &= \sum_k |\partial_\tau \varphi_{\mathcal{A}, \vec{k}}|^2 + |\partial_\tau \varphi_{\mathcal{B}, \vec{k}}|^2 + |k^z \varphi_{\mathcal{A}, \vec{k}}|^2 + |k^z \varphi_{\mathcal{B}, \vec{k}}|^2 \\ &+ ga k^m |\varphi_{\mathcal{A}, \vec{k}} + \varphi_{\mathcal{B}, \vec{k}}|^2 + gb k^m |\varphi_{\mathcal{A}, \vec{k}} - \varphi_{\mathcal{B}, \vec{k}}|^2. \end{aligned} \quad (21)$$

Both \mathcal{A} and \mathcal{B} are free boson theories, and the coupling between them is uniform in the d -dimensional space. Here a and b are both dimensionless constants, and g is a

small coupling constant. By adjusting number m , we can tune the scaling dimension of g : $\Delta = 2z - m$. Since the entire Lagrangian Eq. 21 is quadratic, the entanglement entropy S_A can be calculated exactly.

The entanglement entropy S_A can be calculated in the same formalism as Ref.⁸, where a marginal coupling between two Luttinger liquids was considered ($z = 1$, $m = 2$). Since the exact ground state wavefunctional of field φ_A and φ_B can be written down exactly, one can directly calculate the entropy with the exact reduced density matrix. The coupling in Eq. 21 is uniform in space, it only couples φ_A and φ_B modes with the same momentum, hence the entropy is a simple sum of the entropy of coupled harmonic oscillators at each momentum k . Since the exact result is in general a complicated function of g , we will focus on the leading term after expanding the exact result with small g .

The leading order results with different choices of Δ are summarized as follows:

$$\begin{aligned}
(1), \quad \Delta = d/2, \\
S_A \sim g^2 \log\left(\frac{1}{|g|}\right) L^d; \\
(2), \quad \Delta > d/2, \\
S_A \sim g^{d/\Delta} L^d, \\
(3), \quad \Delta < d/2, \\
S_A \sim g^2 L^d. \tag{22}
\end{aligned}$$

Notice that $d/2$ is the critical value of Δ , when $\Delta < d/2$ the leading term of the Rényi entropy is always $g^2 L^d$, whether g is relevant or not. When $\Delta = d/2$, the leading order entropy acquires a logarithmic correction. In Eq. 22, we have assumed that the infrared cut-off of the system is g instead of $1/L$, *i.e.* $L > g^{-1/\Delta}$. If $L < g^{-1/\Delta}$, the argument of the logarithmic function in Eq. 22 is replaced by L .

It is known that the subleading correction to the boundary law of the ordinary entanglement entropy contains important information about the CFT^{9,10}. In fact, in Eq. 22, in addition to the leading term proportional to the volume of the system, there are also subleading terms. The subleading terms can be calculated conveniently for free boson theory Eq. 21 at $d = 1$, and we summarize our results here:

$$\begin{aligned}
S_{\text{subleading}} &\sim \log\left(\frac{(\sqrt{a} + \sqrt{b})^4}{16ab}\right), \quad \Delta > 0, \\
S_{\text{subleading}} &\sim \Delta \log(L), \quad b = 0, \quad \Delta > 0, \\
S_{\text{subleading}} &\sim g^2, \quad \Delta = 0. \tag{23}
\end{aligned}$$

When $\Delta < 0$, there is no subleading term at order $O(L^0)$.

Interestingly, when the coupling is relevant *i.e.* $\Delta > 0$, the most generic subleading contribution to the entropy

approaches a constant when $g \rightarrow 0$, as long as we take the limit $L \rightarrow \infty$ first. If $b = 0$ (*i.e.* the coupling only affects mode $\varphi_A + \varphi_B$, while $\varphi_A - \varphi_B$ is still gapless), $S_{\text{subleading}}$ is logarithmic of the system size, and its coefficient is a universal constant proportional to the scaling dimension Δ , but it is independent of the magnitude of g .

The logarithmic subleading term in Eq. 23 may have generalizations to other CFTs in one dimension. The universal coefficient of the logarithmic term might be related to the central charge of the CFT. We will study this general theory in future.

B. Perturbative calculation

We can also apply our perturbative formalism Eq. 8 to the coupled CFTs. In the momentum and frequency space, the entropy is evaluated as

$$\begin{aligned}
\log(\text{tr} \rho_A^2) &= \lim_{\beta \rightarrow +\infty} (S'_1 - S'_2), \\
S'_1 &\sim g^2 \sum_{\omega_A, \omega_B} \sum_{k_i, k_{A,j}, k_{B,j}} \frac{4 L^{2h-2d}}{\beta^2 (\omega_A - \omega_B)^2} \\
&\times \frac{1}{(\omega_A^2 + \sum_{i=1}^h k_i^2 + \sum_{j=h+1}^d k_{A,j}^2)^{\frac{1}{2}(d+1-2\Delta_A)}} \\
&\times \frac{1}{(\omega_B^2 + \sum_{i=1}^h k_i^2 + \sum_{j=h+1}^d k_{B,j}^2)^{\frac{1}{2}(d+1-2\Delta_B)}}; \\
S'_2 &\sim g^2 \sum_{\omega} \sum_{k_i, k_{A,j}, k_{B,j}} L^{2h-2d} \\
&\times \frac{1}{(\omega^2 + \sum_{i=1}^h k_i^2 + \sum_{j=h+1}^d k_{A,j}^2)^{\frac{1}{2}(d+1-2\Delta_A)}} \\
&\times \frac{1}{(\omega^2 + \sum_{i=1}^h k_i^2 + \sum_{j=h+1}^d k_{B,j}^2)^{\frac{1}{2}(d+1-2\Delta_B)}} \tag{24}
\end{aligned}$$

We have taken $z = 1$ as example. $\Delta_{A(B)}$ is the scaling dimension of $O_{A(B)}$. In this equation,

$$\omega_A = \frac{2\pi(m+1/2)}{\beta}, \quad \omega, \omega_B = \frac{2\pi n}{\beta}. \tag{25}$$

Correct evaluation of the summation in Eq. 24 will lead to the results consistent with the exact results Eq. 22. In this calculation, one should always take the limit $\beta \rightarrow \infty$ before the limit $L \rightarrow \infty$. When $h = d$ (two CFTs are coupled uniformly in the entire space), the leading contribution to Eq. 24 is

$$S_A \sim g^2 L^d \int \frac{1}{k^{2d+2-2\Delta_A-2\Delta_B}} d^d k. \tag{26}$$

The scaling dimension of g is $\Delta = d + 1 - \Delta_A - \Delta_B$. If we take $\Delta = d/2$, this integral gains a logarithmic contribution. Since $\Delta > 0$, the higher order perturbation will acquire stronger and stronger infrared divergence. If $L < g^{-1/\Delta}$, this logarithmic divergence is

cut-off by $1/L$; if $L > g^{-1/\Delta}$, we expect the summation of the perturbation series will eventually be cut-off by length scale $g^{-1/\Delta}$, so the final answer should be $S_{\mathcal{A}} \sim g^2 \log(1/|g|) L^d$, which is consistent with Eq. 22.

Similarly, if we take $0 < h < d$, then when $\Delta = h/2$ the perturbation theory gives the logarithmic term

$$S_{\mathcal{A}} \sim g^2 \log\left(\frac{1}{|g|}\right) L^h, \quad (\text{when } \Delta = h/2). \quad (27)$$

Now suppose we take $h = 0$ (point contact), the scaling dimension of g is $\Delta = 1 - \Delta_{\mathcal{A}} - \Delta_{\mathcal{B}}$. For simplicity, we assume that $\Delta_{\mathcal{A}} = \Delta_{\mathcal{B}} = \frac{1}{2}(1 - \Delta)$. Then Eq. 24 is evaluated as

$$\begin{aligned} S_{\mathcal{A}} &\sim g^2 \int \frac{1}{k_1^{d+\Delta-1} k_2^{d+\Delta-1}} \frac{1}{(|k_1| + |k_2|)^2} d^d k_1 d^d k_2 \\ &\sim g^2 \int \frac{1}{k^{2\Delta+1}} dk. \end{aligned} \quad (28)$$

When $\Delta < 0$, the point contact is irrelevant, and this integral gives a constant result $S_{\mathcal{A}} \sim g^2$, with a nonuniversal coefficient. When $\Delta = 0$ (marginal point contact), we obtain a logarithmic contribution, regardless of the total dimension d :

$$S_{\mathcal{A}} \sim g^2 \log(L). \quad (29)$$

In this case, since the point contact is marginal, the higher order perturbation has the same logarithmic divergence as the leading order, hence we expect this logarithmic divergence persists even after we sum the entire series, and it can only be cut-off by L .

C. Scaling Argument

In this section we will try to understand the results obtained in the previous two subsections using a simple scaling argument. The argument in this section is a generalization of Ref.¹¹, where a scaling argument was introduced to explain the logarithmic contribution of the ordinary entanglement entropy of 1d CFT. Scaling argument was also introduced to understand the entanglement entropy close to finite temperature critical points¹².

We first note that, if the two coupled systems are gapped, the Rényi entropy is obviously proportional to the volume of the coupled submanifold, and it should scale as g^2 for weak coupling: $S_{\mathcal{A}} \sim g^2 L^h$. This is because $S_{\mathcal{A}}$ is positive definite, hence the leading contribution should be an even function of g . Also, the entanglement entropy vanishes in the decoupling limit $g \rightarrow 0$. Thus the leading analytic contribution from the coupling should be g^2 .

Now we consider two CFTs coupled on a h -dimensional submanifold \mathcal{M} of the space, and the entanglement entropy is collected while coarse-graining the system. At length scale l , the size of the

coupled subsystem is effectively L^h/l^h . Within length scale interval $d(\log l)$, the entanglement entropy is

$$d(S_{\mathcal{A}}) \sim \frac{L^h}{l^h} g_l^2 d(\log l). \quad (30)$$

g_l is the effective coupling constant at length scale l , and as long as g_l is weak, $g_l \sim g l^{\Delta}$. Thus if $\Delta > 0$, the total entropy is

$$S_{\mathcal{A}} \sim \int_{l=a}^{l=\text{Min}[|g|^{-1/\Delta}, L]} d(\log l) \frac{L^h}{l^h} g^2 l^{2\Delta}. \quad (31)$$

This integral gives us the following results for general $d \geq h > 0$, with different choices of Δ :

$$\begin{aligned} (1), \quad \Delta &= h/2, \\ S_{\mathcal{A}} &\sim g^2 \log\left(\frac{1}{|g|}\right) L^h, \quad L \gg g^{-1/\Delta}; \\ S_{\mathcal{A}} &\sim g^2 L^h \log(L), \quad L \ll g^{-1/\Delta}; \\ (2), \quad \Delta &> h/2, \\ S_{\mathcal{A}} &\sim g^{h/\Delta} L^h, \quad L \gg g^{-1/\Delta}; \\ S_{\mathcal{A}} &\sim g^2 L^{2\Delta}, \quad L \ll g^{-1/\Delta}; \\ (3), \quad \Delta &< h/2, \\ S_{\mathcal{A}} &\sim g^2 L^h. \end{aligned} \quad (32)$$

Now the critical value of Δ becomes $h/2$. When $h = d$, and $L > g^{-1/\Delta}$, Eq. 32 reduces to the results Eq. 22 obtained from exact calculations.

In the special case with $h = 0$, *i.e.* the CFTs are coupled through a point contact, the integral in Eq. 31 gives us the following results:

$$\begin{aligned} \Delta = 0, \quad S_{\mathcal{A}} &\sim g^2 \log(L), \\ \Delta > 0, \quad S_{\mathcal{A}} &\sim C, \quad L \gg g^{-1/\Delta}, \\ S_{\mathcal{A}} &\sim g^2 L^{2\Delta}, \quad L \ll g^{-1/\Delta}, \\ \Delta < 0, \quad S_{\mathcal{A}} &\sim g^2. \end{aligned} \quad (33)$$

C is a constant which does not scale with g . When $\Delta = 0$, namely the case with a marginal point contact, the leading contribution to the entanglement entropy is a logarithmic term, which is independent of the spatial dimension d . This conclusion confirms our calculation in the previous subsection (Eq. 28, Eq. 29), and confirms the calculation for Fermi liquid with a point contact (Eq. 18).

This simple scaling argument should be precise if g is irrelevant or marginal, for arbitrary d and h . If g is relevant, the integral of length scale in Eq. 31 was taken only from a (lattice constant) to the scale where g becomes nonperturbative. It seems like we have ignored

the entropy contribution *after* g becomes nonperturbative. To understand this problem, we need to know the long wavelength properties of the system when g is relevant, and there are two possibilities:

1. In most cases, a relevant coupling g opens up a gap (or local gap) for CFTs \mathcal{A} and \mathcal{B} , namely all the correlation functions $G(\tau_1 - \tau_2, x_1 - x_2)$ with $x_1, x_2 \in \mathcal{M}$ decays exponentially when $|\tau_1 - \tau_2| \rightarrow \infty$. In this case, the system can be driven into either a direct product state between \mathcal{A} and \mathcal{B} , or a maximally or partially entangled state between \mathcal{A} and \mathcal{B} with a saturated entanglement. For instance, if \mathcal{A} and \mathcal{B} are free boson fields, and in the submanifold \mathcal{M} they are coupled as $-ag(\varphi_{\mathcal{A}} + \varphi_{\mathcal{B}})^2 - bg(\varphi_{\mathcal{A}} - \varphi_{\mathcal{B}})^2$, then as long as $ag > 0$ and $bg > 0$, the relevant coupling g will keep $\varphi_{\mathcal{A}} = \varphi_{\mathcal{B}} = 0$ in \mathcal{M} , hence the system becomes a direct product state between \mathcal{A} and \mathcal{B} in the long wavelength limit. Then the entropy with length scale $l > g^{-1/\Delta}$ is ignorable, and there is no correction to Eq. 31.

When g drives the system into a maximally or partially entangled state, then we need to add another contribution to the entropy, which is

$$S'_{\mathcal{A}} \sim \frac{L^h}{(g^{-1/\Delta})^h} = g^{h/\Delta} L^h. \quad (34)$$

With this extra contribution, our results in Eq. 32 and Eq. 33 still hold.

2. If some of the correlation functions in the coupled submanifold \mathcal{M} remain power-law even with relevant g , then there is a residual scaling invariance after g becomes nonperturbative. In this case, we need to include the following extra contribution to Eq. 31:

$$S'_{\mathcal{A}} \sim \int_{l=g^{-1/\Delta}}^{l=L} d(\log l) \frac{L^h}{l^h}. \quad (35)$$

This integral does not modify any of the leading order terms in Eq. 32 with $h > 0$, but it leads to a logarithmic contribution to Eq. 33 with $\Delta > 0$ and $h = 0$, *i.e.* it only affects the case with a relevant point contact coupling

between \mathcal{A} and \mathcal{B} . For instance, in Eq. 21, although the mode $\varphi_{\mathcal{A}} - \varphi_{\mathcal{B}}$ remains gapless when $b = 0$, the exact results Eq. 22 always agree with Eq. 32 obtained from scaling integral Eq. 31 for $h = d$, no matter $b = 0$ or not.

V. SUMMARIES AND EXTENSIONS

In this work, we studied the entanglement entropy of coupled Fermi liquids and CFTs. Three different methods were used for the calculation: perturbation theory, scaling argument, and exact ground state wavefunctional. These three approaches are consistent with each other for all the cases that we can check.

It has been demonstrated that the holographic method is a very powerful way of calculating the entanglement entropy^{9,13} of CFT, assuming there is a bulk AdS space duality of the boundary CFT. The ordinary entanglement entropy is related to the area of the minimal surface of the bulk AdS space. In future, we will try to develop a holographic formalism to produce the results in the current paper. Since AdS/CFT duality effectively “geometrizes” the RG flow at the boundary CFT theory, we expect the holographic calculation of the entanglement entropy to be qualitatively equivalent to the scaling argument discussed in this paper.

Significant progresses have been made in numerical simulation of quantum many-body states. For instance, the multi-scale entanglement renormalization ansatz (MERA) is especially powerful in simulating one dimensional CFT^{14,15}. In future, it will also be interesting to verify the conclusions in our paper numerically.

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